End Correction at a Flue Pipe Mouth

by Johan Liljencrants

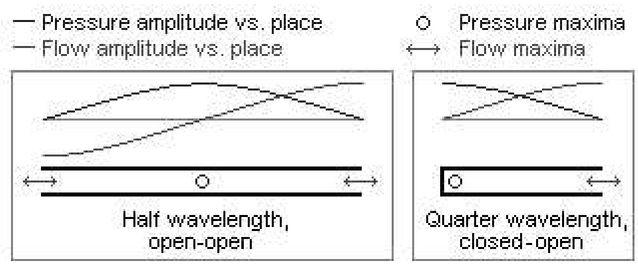
Abstract

A simple notion is that the fundamental resonance of a pipe occurs when the sound wavelength is half or a quarter of the resonator length. It is however well recognized that the practical frequency comes out lower than this, you have to apply an end correction, the pipe appears to be acoustically somewhat longer than its physical length. A formula for the basic mechanism behind this is theoretically derived, then expanded into the case where the open end area is made smaller than the pipe cross section.

The end correction was experimentally determined for several pipes with mouths extending 360 and 90 degrees of the circumference. Formulas are given to compute the end correction, using optimal coefficients found from these measurements.

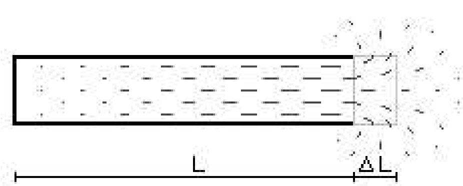
Introduction

The pitch f of flue instruments like the flute, organ pipe, or whistle is predominantly controlled by a resonator length L. This length is closely connected to the wavelength $\lambda = c/f$, where c is the speed of sound. Examples of the most common cases are open and stoppered organ pipes, commonly characterized by their working length as being half and quarter wavelength, respectively. When such pipes oscillate at their fundamental pitch their internal standing wave patterns of acoustic pressure and flow are classically illustrated this simplified way.



Let us initially inspect a quarter wavelength resonator tube of length L and area A, closed at one end. This tube has a total acoustical mass $M^* = \rho L/A$ and acoustical capacitance $C^* = AL/(\rho c^2)$, where ρ is density of the air and c is its speed of sound. The effective resonating mass and capacitance are less than these total ones. They are reduced by the factor $2/\pi$ due to the sinusoidal distribution within the tube of flow and pressure. (This factor is the area of a sine quadrant, divided by the area of its circumscribed rectangle). Thus the effective mass is $M = 2\rho L/(\pi A)$ and effective capacitance $C = 2AL/(\pi \rho c^2)$. Inserting these into the resonance formula $f = 1/(2\pi\sqrt{MC})$ and cleaning up, leads into the well known quarter wave expression $f = c/(4L)$ for the fundamental resonance.

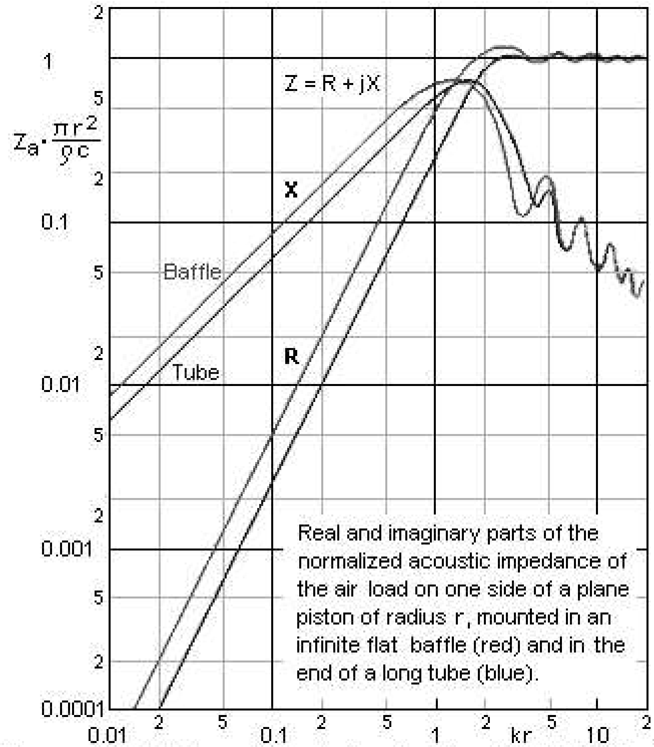
The flue end of a pipe where the air jet blows across the mouth is an open end of the resonator, having an acoustic flow maximum. The acoustic pressure at this place in principle is minimum. Still it is far from zero, it is the sound pressure you can hear here. Inside the resonator the sound pressure is considerably higher, its magnitude approaches the blowing pressure.



The air immediately outside the end of the pipe takes part in the acoustic oscillation. This air makes the pipe appear to be acoustically somewhat longer than its physical length. This apparent length increase ΔL is called the end correction. To compute the resonance frequency the length measure one should use is the sum of physical length plus the end correction.

Theory

A theoretical basis for computation of the end correction is the 'radiation acoustic impedance of a circular piston', reproduced here. This impedance tells the ratio of acoustic pressure at the piston, divided by the flow rate induced by it. The piston does not physically exist, it is an abstract theoretical vehicle to state that one assumes the air speed to be the same at all places across the tube end. This is a good approximation, but not exactly true in reality, since air viscosity reduces the flow rate in the boundary layer very close to the tube surface.



Frequency is plotted on a normalized scale, where $kr=2\pi fr/c=2\pi r/\lambda$.
After Beranek: Acoustics, McGraw-Hill 1954, pp 119, 122.

Two different cases are illustrated here. In blue for a free tube end and in red for a baffled tube, i.e. when a wall limits the external sound to spread only into a half space rather than all around.

There is a critical frequency, typically taken as $kr=1$, which implies that wavelength equals the circumference of the piston/tube. At higher frequencies, or greater r, the tube end tends to impedance match the ambience such that the tube does not act as a resonator, but just a transmission line. The scale of whistles and organ pipes is always such that kr is substantially less than unity, so what applies to our problem is the left half of the diagram.

The impedance Z is composed from two parts, the real resistance R and the imaginary reactance X.

R tells the in-phase component of the pressure to flow ratio. A given flow U will develop a power $W=RU^2$ that is lost from the resonator and is radiated into the ambient space to become a useful sound. Normally this 'radiation resistance' is a major determinant of the resonator Q value.

The reactance $X=2\pi f M$ is the quadrature component. The nice thing is that X turns out proportional to frequency for $kr<1$, hence its corresponding acoustical mass M is constant within this $kr<1$ range. This M is the effective measure of the co-oscillating air outside the tube opening. From the upward sloping blue X contour in the diagram, leaving out trivial intermediate formula manipulations, we arrive at the most commonly cited expression

$\Delta L = 0.6 * r = 0.3 * D$, where the equivalent tube radius is $r=\sqrt{A/\pi}$.

Since it is rather immaterial whether the tube is circular, quadratic, or even moderately oblong rectangular, let us here stick to the correction expressed in terms of the tube area, such that it amounts to

$\Delta L = 0.34\sqrt{A}$

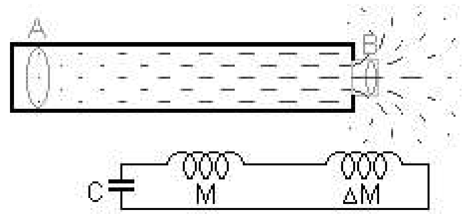
For a baffled tube end we similarly find a slightly larger value, namely 0.48 times the square root of tube area.

End correction for a mouth

Now let us narrow the open end of the tube with an aperture, into a mouth area B, less than or equal to the tube area A. This means that we effectively remove the classical end correction and instead add the acoustic mass of a tube of area B. The physical length of this tube is essentially zero, but we have to add a new end correction for this one, indeed for both sides of its aperture. On the outside we can simply assume the same formula as before. Similarly might apply also to the inside of the aperture, but only when B is appreciably smaller than the tube A. With a growing aperture area such an internal end correction decreases to ultimately vanish when B reaches A. There are plausible models for this, but for the present we neglect any internal correction.

There is a paradox in that the basic correction formula for the B aperture alone gives a length that is smaller than for the original A. This might suggest the resonance frequency to rise because of the constriction, contrary to all observation. To resolve this we must go back to the circuit elements of the resonator. The capacitive element C (the elastic action of the resonator interior) is like before, unaffected by the aperture. But the inertive element M of the tube is augmented with the ΔM that comes in the aperture. When we compare the two as given above we get

$$\frac{\Delta M}{M} = \frac{\rho\, 0.34\sqrt{B}/B}{2\rho L/(\pi A)} = \frac{0.53\, A}{L\sqrt{B}}$$



The paradox is resolved by the fact, that when the area *B* is made smaller, then the additive ΔM becomes larger. Recognizing that for small deviations from 1, the square root in the frequency formula functions like $\sqrt{1+\varepsilon} \approx 1+\varepsilon/2$, we then finally arrive at

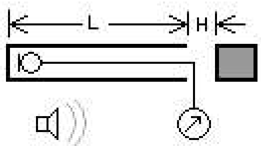
$$\Delta L = \frac{0.27\, A}{\sqrt{B}}$$

We would have appreciated when the coefficient in this result had rather been 0.34 , such that it complied with the original formula when $B=A$. But when we assume the mouth aperture to be partially baffled, that would correspond to a credible increase in the coefficient.

The importance of this theoretical derivation is to suggest that the end correction is basically proportional to tube area over square root of mouth area. It remains to find practical values for the proportionality factor, and how to specify resonator physical length.

The basic $A/\sqrt{B}$ relation was used as a direct value for the end correction by Ising (1969), apparently adopted from Ingerslev and Frobenius (1947).

Experimental measurements



A number of tubes with one end closed were examined for their passive fundamental resonances. The open ends were adjustable for variation in cutup H.

The system was excited by an external loudspeaker, driven from a tone generator, and frequency was monitored with a counter to 0.1 Hz accuracy. A midget microphone was placed inside the resonator such that the resonance peak could be located by adjusting frequency. Having found the -3 dB points f_a and f_b of the resonance peak, then the resonance is found as $f_1=(f_a+f_b)/2$ while the quality factor $Q=f_1/(f_b-f_a)$.

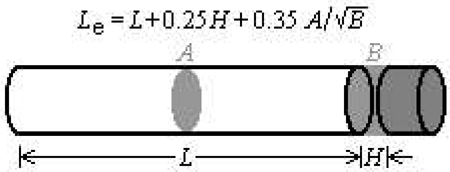
Knowing f_1 and sound speed (ambient temperature should be accounted for) the actual quarter wavelength is computed. This then equals the physical resonator length *L* plus a sought, experimentally determined end correction. This is compared to the theoretical model, the right member of the expression

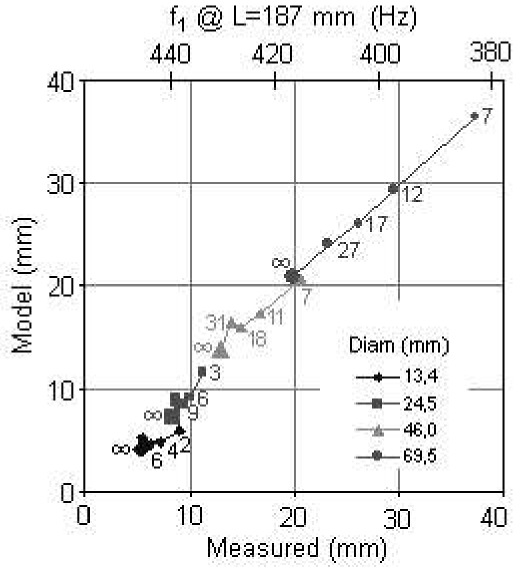
$$\frac{\lambda}{4} = \frac{c}{4f_1} = L + \Delta L = L + \alpha H + \beta A/\sqrt{B}$$

The various experimental values were collected in an Excel spreadsheet. A final step is to find values for the coefficients α and β, by trial and error, such that the model values (computed from A, B, and H) come maximally close to the experimental ones.

Cylindrical whistle, 360 degree mouth

A set of cylindrical tubes were used, where length *L* from stopper to end was in all cases set to 187 mm, corresponding to resonance in the 400 Hz range. The internal diameters were 13.4, 24.5, 46.0, and 69.5 mm, wall thickness between 1.5 and 2 mm. For each, a block of same outer diameter was placed at variable distances *H* from the open end in order to simulate the pipe languid and foot.





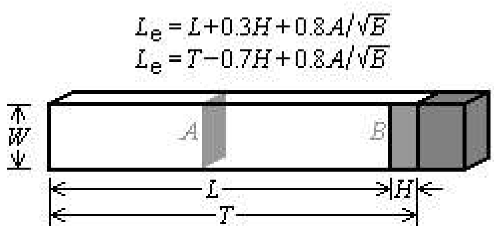
The two coefficients α and β for H and $A/\sqrt{B}$ shown here are empirically adjusted for a best match between theory and experiment, using the same values for all tube diameters. The data points line up reasonably well along the diagonal in this plot of measured vs. predicted end corrections. The RMS error between data and model is 1.2 mm. Optimizing for a single tube diameter, the match may improve using slightly different coefficients. But including the tube diameter as another parameter in the optimization did not improve the match substantially.

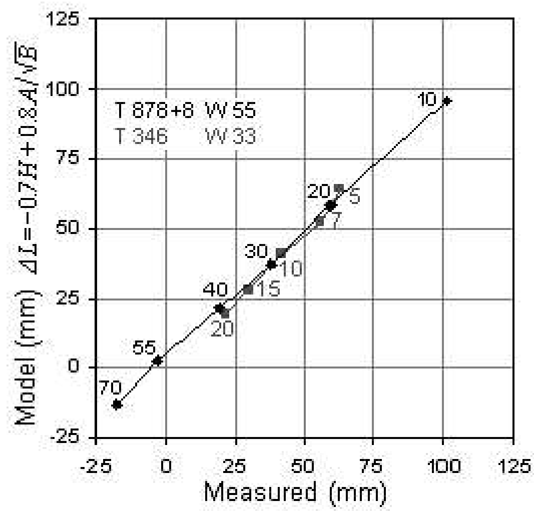
The small colored figures tell values of H. Here resonator length is taken as the tube proper, such that the case of infinite cutup can be included. But common practise is otherwise to define the resonator length as the sum of L and H.

It is interesting to note the additional term with the cutup H. This apparently accounts for an extra distance for the sound wave to travel, from the end of the tube to an effective center of the mouth opening. This may come from that the mouth is located at the side of the tube rather than centrally at its end, as was postulated in the theoretical model.

Square organ pipe, mouth on one side

The following data were obtained from two regular square stoppered wooden organ pipes where the cutup could be varied by moving the labium. This means that the front plate length L differed between data points, while T was constant.





Because this graph and formula is based on the total length T, the end correction becomes negative at sufficiently large H.

Here the common way, defining the resonator length as $T = L+H$ is shown in the lower formula and the graph, while the upper formula uses the front plate length L. The two formula alternatives render identical results for the effective length L_e, which is the quarter wavelength at fundamental resonance.

These two pipes render the same optimal coefficients once the length T of the bigger one was artificially increased by 8 mm. This extra correction is the thickness of its pipe walls, apparently introducing an extra length or a baffling effect. Here the RMS error between data and model is 1.7 mm.

The most obvious difference against the previous 360 degree mouth is that the coefficient for the $A/\sqrt{B}$ term is now much bigger, supposedly because the external sides of the pipe act as baffles.

Conclusions

The foundation for the mouth end correction was theoretically established as having a factor $A/\sqrt{B}$ where A and B are the areas of resonator and mouth. This was verified by resonance measurement on a number of different pipes. In matching the end correction model to the experimental data it was found that the weight of this factor differs very much depending on the rotational extent of the mouth opening, here 90 and 360 degrees were examined. For a mouth at the side of the resonator tube also a fraction of the cutup height should be added to the end correction.

It must be noted that the fundamental speaking frequency of the pipe when blown is not precisely the same as the resonance frequency studied here. When blown, the pipe adjusts to whatever phase angle is imposed by the flue exciting mechanism, most probably characterized by its Ising intonation number. This, in turn, is determined from cut up, flue airband thickness, and blowing pressure.

References

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Ingerslev F, Frobenius W (1947): Trans. of the Danish Acad. of Technical Science, No 1.
Ising H (1969): Über die Klangerzeugung in Orgelpfeifen. Diss.,Technisches Universität Berlin.

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